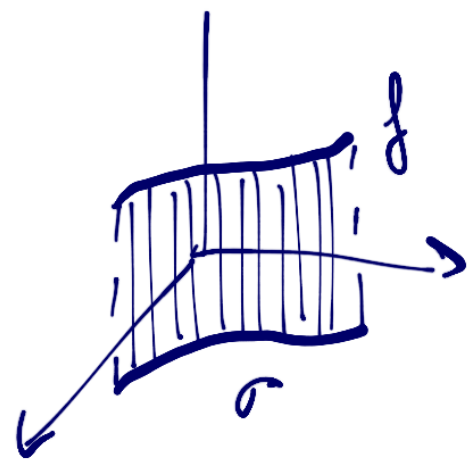
Line integrals

$\int_\sigma f \equiv$ average of density f along the trajectory
line
path

$f: \mathbb{R}^n \to \mathbb{R}$

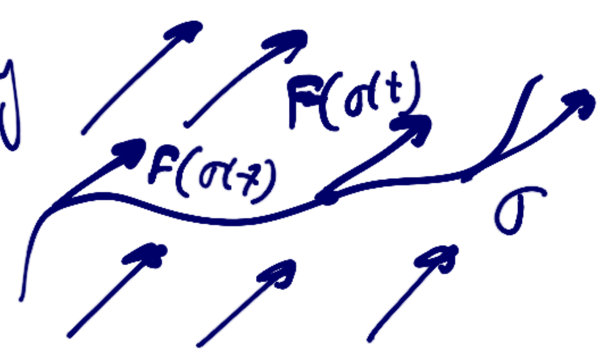
$\sigma \equiv$ trajectory.



$\int_\sigma F \equiv$ represents how a particle moves along the trajectory under the action of F

$F: \mathbb{R}^n \to \mathbb{R}^n$ vector field.

$\sigma \equiv$ trajectory

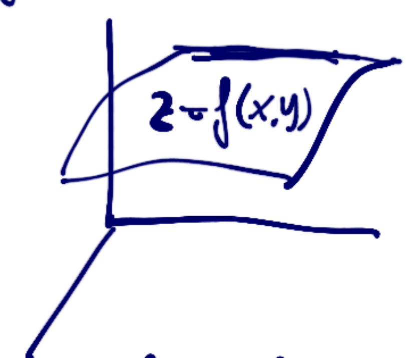


Surface integrals

Integrals of scalar and vector fields on surfaces.

Surface could be the graph of a scalar function.

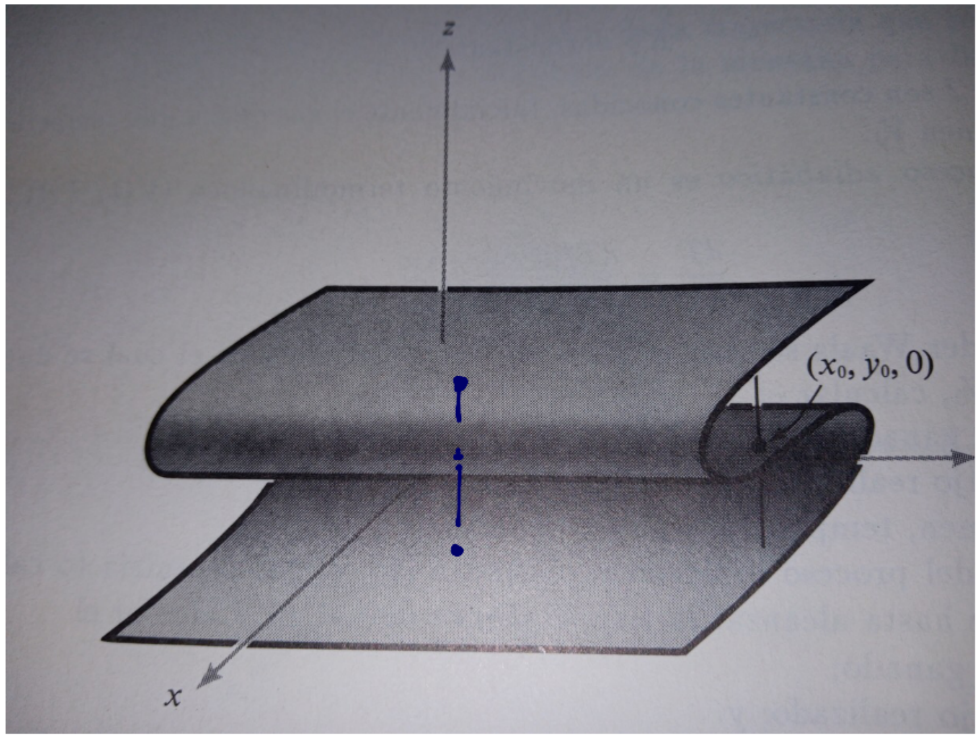
$$z = f(x,y)$$



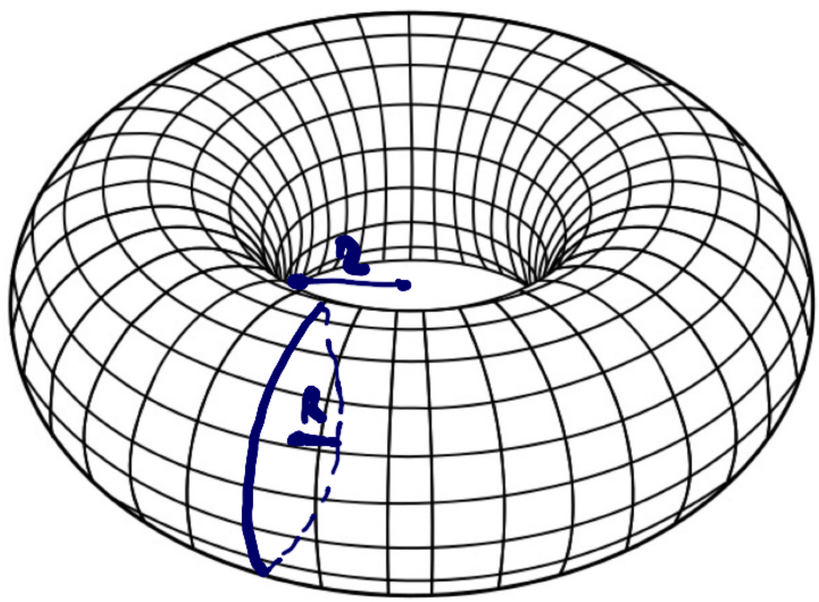
we will consider more general surfaces.
not only the graph of function.

$x - z + z^3 = 0$

(Bended paper)



Famous Torus



$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2$$

a function / as a parametrization

$\not\equiv$

its image (geometrical object)

Definition

A parametrised surface is a continuous function

$$\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

with $D \equiv$ parametrical domain in $\mathbb{R}^2$

Then, the surface S is the image of the function Φ, $S = \Phi(D)$ and we can write

$$\Phi(u,v) = \Big(x(u,v), y(u,v), z(u,v) \Big)$$

Remark

- $S = \Phi(D)$ is simple if the function Φ is one-to-one.

- Φ = parametrization. is differentiable or of class C^1, then the surface is C^1

(smooth surface)

Example: Plane $x+y+z=1$: Implicit form/surface

$$\left.\begin{aligned} x(u,v) &= x = u \\ y(u,v) &= y = v \\ z(u,v) &= z = 1-u-v \end{aligned}\right\} \text{ Parametrisation of the surface.}$$

$$\Phi(u,v) = (u, v, 1-u-v)$$

Definition

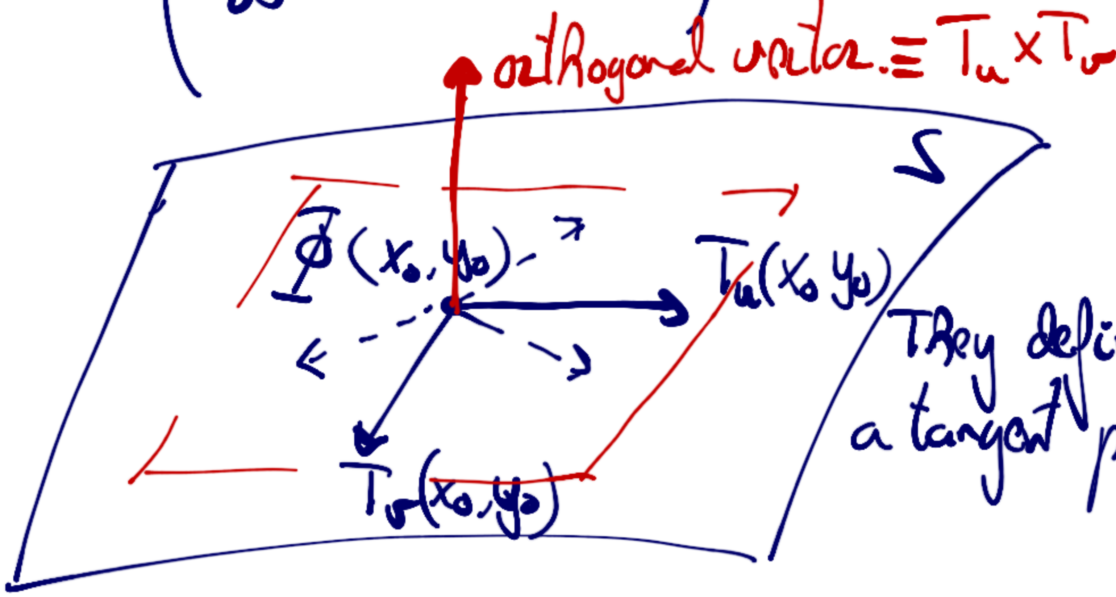
For a parametrised surface $S = \Phi(D)$, smooth so Φ diff. at any point in D

we define the tangent vectors or tensors.

$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = \frac{\partial \Phi}{\partial u}$$

and

$$T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = \frac{\partial \Phi}{\partial v}$$



They define a tangent plane.

Definition

Let $\Phi : D \longrightarrow \mathbb{R}^3$ be a regular/smooth surface

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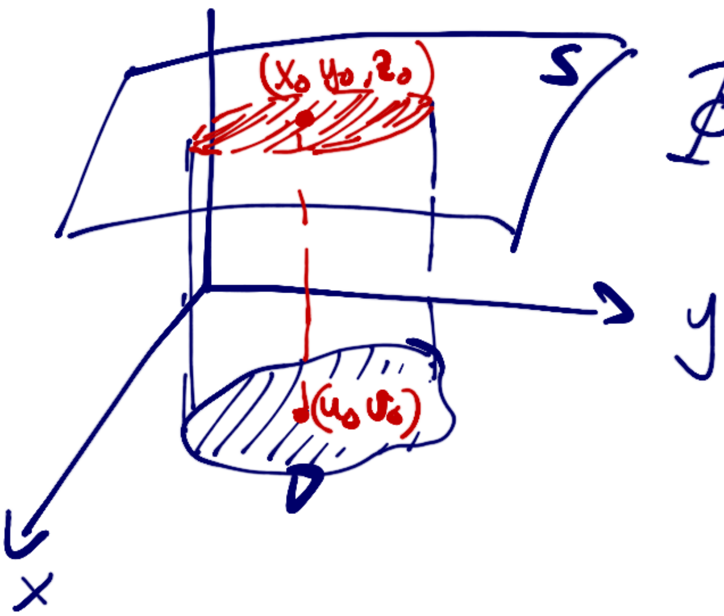
$$S = \Phi(D) \qquad \phi \in C^1$$

and $T_u \times T_v \neq 0$ at any point in D



$$\Phi(u_0 v_0) = (x_0 y_0, z_0)$$

$$\underbrace{(T_u \times T_v)(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0}_{\text{Tangent plane}}$$

$$n = \frac{T_u \times T_v}{\|T_u \times T_v\|} \quad \text{normal unitary vector.}$$

Example : Parametrise the sphere (Problem 1 i) in 4.3)

$$S = \{(x,y,z) \in \mathbb{R}^3 ,\ x^2+y^2+z^2 = R^2\}$$

Use the function $\Phi : D \longrightarrow \mathbb{R}^3$

$$\Phi(u,v) = \left(\Phi_1(u,v),\ \Phi_2(u,v),\ \Phi_3(u,v)\right)$$

applying spherical coordinates. (with fixed radius R)

$$\left.\begin{aligned} x &= R\cos u \sin v \\ y &= R \sin u \sin v \\ z &= R\cos v \end{aligned}\right\} \qquad \begin{aligned} &D \subset \mathbb{R}^2 \\ &(u,v) \in D \end{aligned}$$

$$\Phi(u,v) = \Big(\underbrace{R\cos u \sin v}_{\Phi_1},\ \underbrace{R\sin u \sin v}_{\Phi_2},\ \underbrace{R\cos v}_{\Phi_3}\Big)$$

$$D = \{(u,v) \in \mathbb{R}^2 ,\ u \in [0, 2\pi] ,\ v \in [0, \pi]\}$$ Parametrical set.

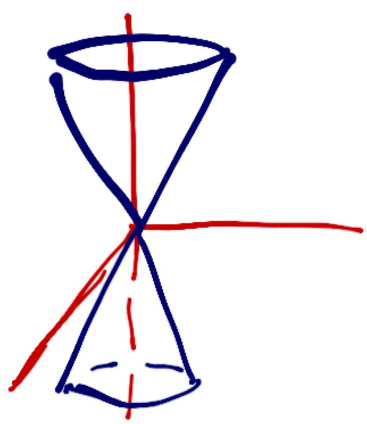
Different parametrization.

$$\Phi_{\pm}(u,v) = \left(u, v, \pm\sqrt{R^2-u^2-v^2}\right)$$ not one-to-one

Example: Problem 1 - ii) in 4.3

Parametrise a cone

$$S = \{(x,y,z) \in \mathbb{R}^3, \; x^2+y^2 = a z^2\}$$

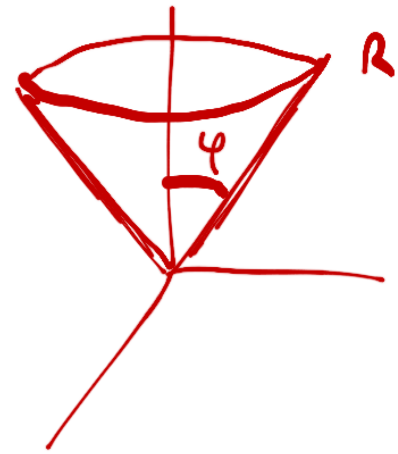


$$\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(u,v) \longrightarrow \Phi(u,v)$$

$$\Phi(u,v) = \left(\Phi_1(u,v), \Phi_2(u,v), \Phi_3(u,v)\right)$$

Use again spherical coordinates.

$$\left.\begin{aligned} x &= \rho \cos u \sin v \\ y &= \rho \sin u \sin v \\ z &= \rho \cos v \end{aligned}\right\}$$



Substituting into the expression of S.

$$\underbrace{x^2+y^2}_{\parallel} = a z^2$$

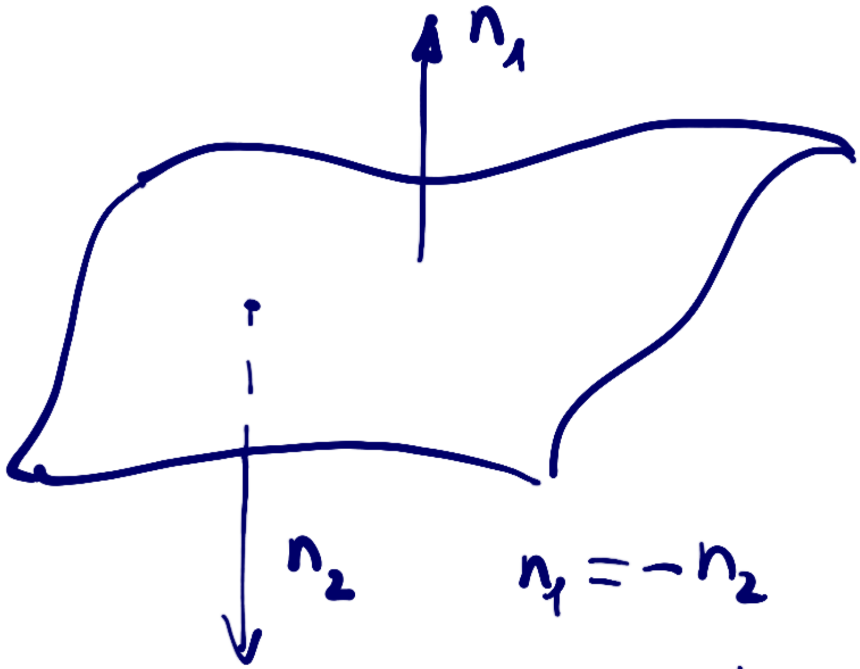
$$\sin^2 v = a \cos^2 v \Rightarrow \underline{\underline{\operatorname{tg} v = \pm\sqrt{a}}}$$

$$v = \pm \operatorname{arctg}\sqrt{a}$$

Then, $\left\{\begin{array}{l} 0 \le \rho \le R \quad \text{(if we fix the height)} \\ 0 \le u \le 2\pi \\ v = \operatorname{arctg}\sqrt{a} \end{array}\right.$

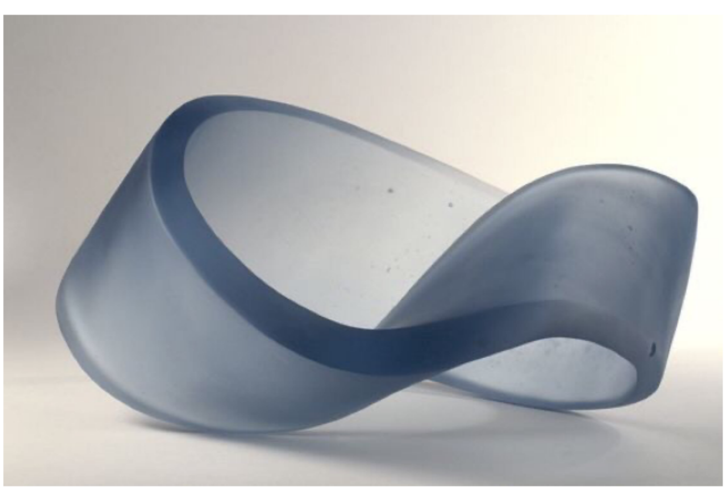
Orientation

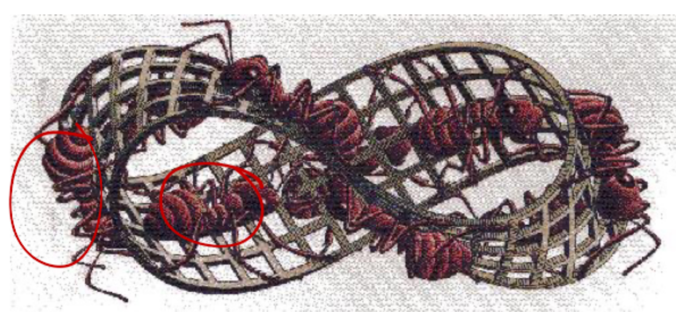
Exterior or interior orientation



Paradigmatic example: **Moebius strip.**

does not have orientation.





Example: $\underbrace{x^2+y^2+z^2=1}_{S \text{ in } \mathbb{R}^3}$: Sphere of radius 1.

Parametrization:

$$\left.\begin{aligned} x &= \cos\theta \sin\varphi \\ y &= \sin\theta \sin\varphi \\ z &= \cos\varphi \end{aligned}\right\} \qquad \begin{aligned} &\Phi : D \to \mathbb{R}^3 \\ &D = \left\{(\theta,\varphi)\in\mathbb{R}^2,\ \begin{aligned}0\le\theta\le 2\pi \\ 0\le\varphi\le\pi\end{aligned}\right\} \end{aligned}$$

$$\Phi(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$$

Tangent vectors.

$$T_\theta = \frac{\partial\Phi}{\partial\theta} = (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0)$$

$$T_\varphi = \frac{\partial\Phi}{\partial\varphi} = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)$$

Orientation is given by the orthogonal vector.

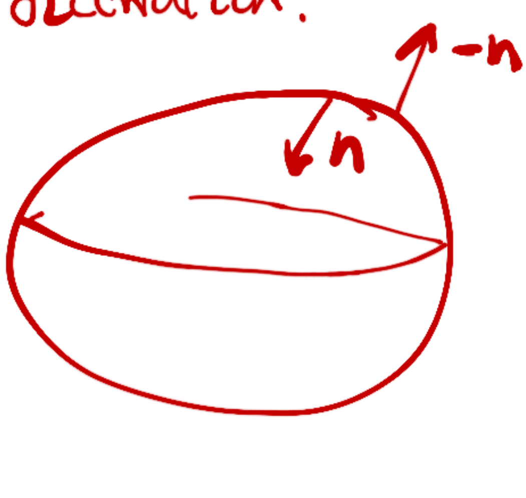
$$T_\theta \times T_\varphi = \begin{vmatrix} i & j & k \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix}$$

$$= -\sin\varphi$$

Since $0 \le \varphi \le \pi$

$$-\sin\varphi \le 0$$

then the normal vector points inward changing the orientation.



Surface Integrals.

<u>Definition</u>

$\underline{\Phi}: D \longrightarrow \mathbb{R}^3$ parametrization of a differentiable surface $S = \underline{\Phi}(D)$ and

$f: \underline{\Phi}(D) \longrightarrow \mathbb{R}$ cont. scalar function

Then

$$\boxed{\int_{\underline{\Phi}} f \cdot dS = \iint_D f(\underline{\Phi}(D)) \cdot \|T_u \times T_v\| \, du \, dv}$$

In particular, the area of a surface S

$$\boxed{A(S) = \int_{\underline{\Phi}} 1 \cdot dS = \iint_D \|T_u \times T_v\| \, du \, dv}$$

average value of f in $S = \Phi(D)$ as

$$\boxed{\frac{1}{A(S)} \int_{\Phi} f \, ds.}$$

Example: $S : x^2+y^2+z^2=1$ Problem 1 i) - 4.3

$$\Phi : \left. \begin{array}{l} x = \cos\theta \sin\varphi \\ y = \sin\theta \sin\varphi \\ z = \cos\varphi \end{array} \right\} \quad D = \{ 0 \le \theta \le 2\pi, \ 0 \le \varphi \le \pi \}$$

$$\Phi : D \longrightarrow \mathbb{R}^3 , \quad \| T_\theta \times T_\varphi \| = \sin\varphi$$

Then

$$A(S) = \iint_D \| T_\theta \times T_\varphi \| \, d\theta \, d\varphi = \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \, d\theta$$

$$= 2\pi \left(-\cos\varphi \right) \Big]_0^{\pi} = 4\pi \quad \left(\begin{array}{l} \text{If } R \neq 1 \\ 4\pi R^2 \end{array} \right)$$

Remark

- $$\int_{\Phi_1} f \cdot dS = \int_{\Phi_2} f \cdot dS$$ two parametrizations Φ_1, Φ_2

- Area of a surface S which is the graph of a function of class C^1

$$z = g(x, y)$$

It might be parametrised by

$$\left.\begin{aligned} x &= u \\ y &= v \\ z &= g(u,v) \end{aligned}\right\} : \Phi$$ then

$$\|T_u \times T_v\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$$

Definition

$\underline{\Phi} : D \longrightarrow \mathbb{R}^3$ diff (of class C^1)

$F : S = \underline{\Phi}(D) \longrightarrow \mathbb{R}^3$ vector field.

$$\int_{\underline{\Phi}} F \cdot dS = \iint_D F(\underline{\Phi}(D)) \cdot \underline{\underline{(T_u \times T_v)}}\, du\, dv$$

However, for two parametrizations $\underline{\Phi}_1$ and $\underline{\Phi}_2$

a) $\displaystyle\int_{\underline{\Phi}_1} F = \int_{\underline{\Phi}_2} F$ with the same orientation

b) $\displaystyle\int_{\underline{\Phi}_1} F = -\int_{\underline{\Phi}_2} F$ with opposite orientation.

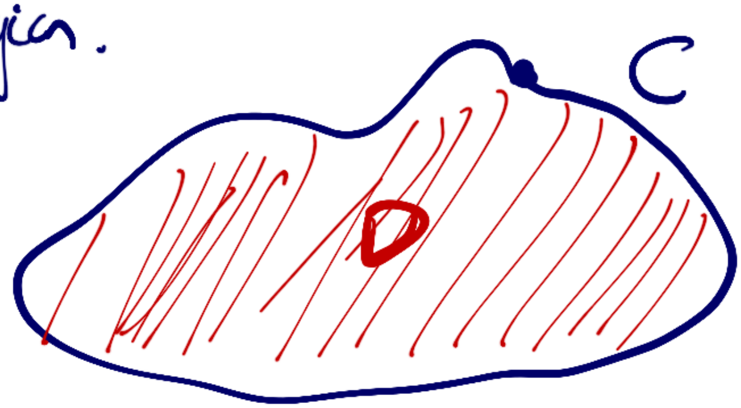
Theorems relating differential vectorial calculus and integral vectorial calculus.

$\Downarrow$

arise in physical problems.

Green's Theorem

It establishes a relation between a line integral along a <u>closed curve</u> and a double integral on the enclosed region.



Green's Theorem (in $\mathbb{R}^2$)

$D \subset \mathbb{R}^2$ open set in the plane

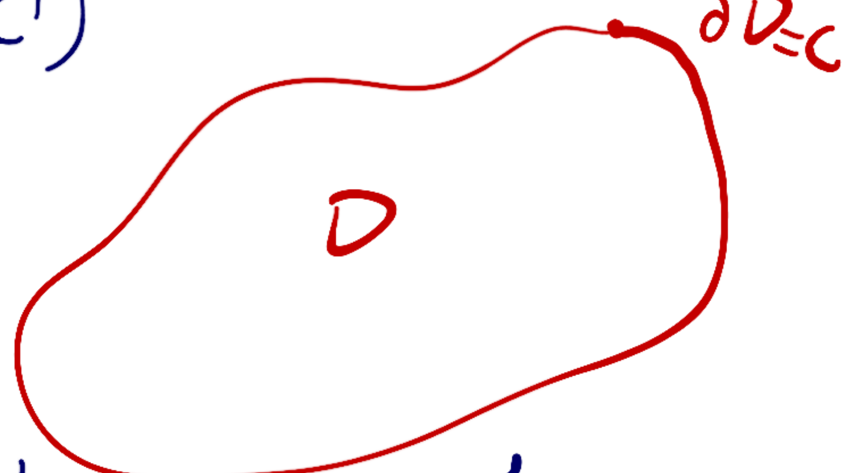
$\partial D \equiv$ boundary of D is closed, simple

$\sigma(a) = \sigma(b)$ no multiple points.

and smooth

(of class C^1)

$\sigma \equiv$ parametric form of $\partial D = C$



P, Q two scalar functions of class C^1

$P, Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Then, if $F(x,y) = \big(\underbrace{P(x,y)}_{F_1(x,y)}, \underbrace{Q(x,y)}_{F_2(x,y)} \big)$ vector field.

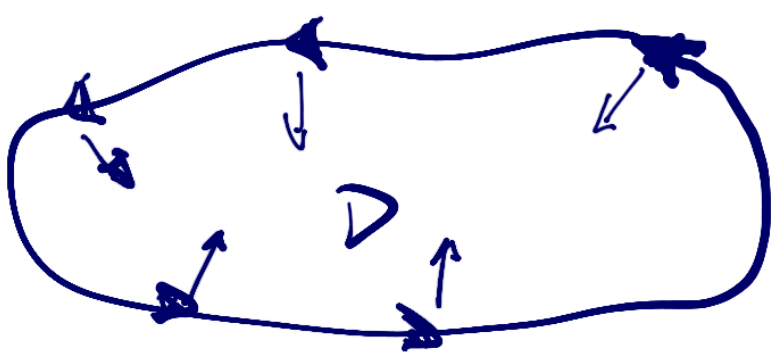
we find that

$$\underbrace{\int_{\partial D} F \cdot dr}_{\text{line integral.}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy}_{\text{double integral.}}$$

Remark

a) Orientation

Positive orientation $\rightsquigarrow$ anticlockwise.

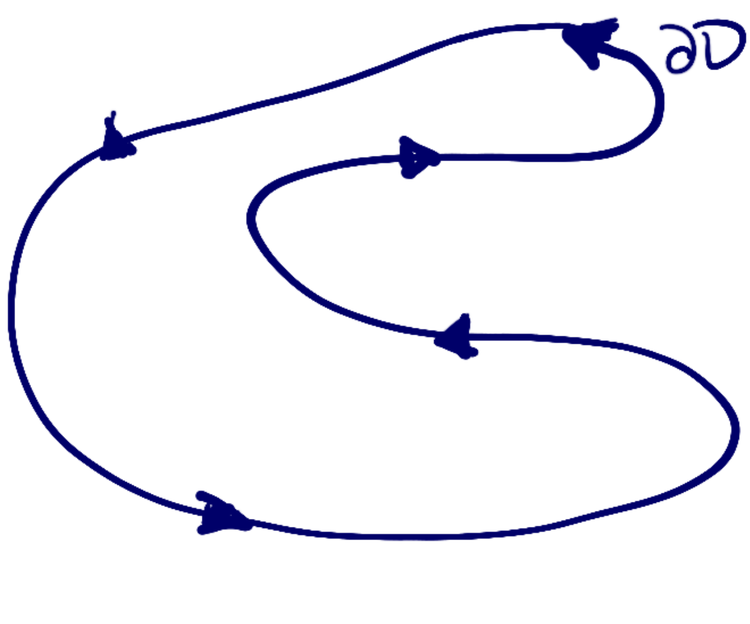


b) Curve must be simple.

σ must be one-to-one.

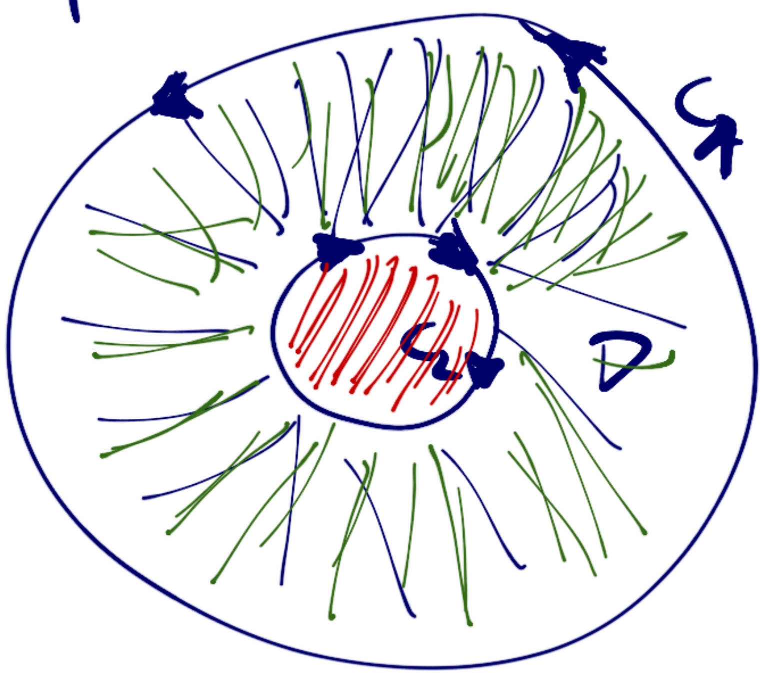
c) Generalisation of Green's theorem

For general regions.



For some regions we must perform some transformations to apply the theorem.

For example an annulus



$C = C_1 \cup C_2$

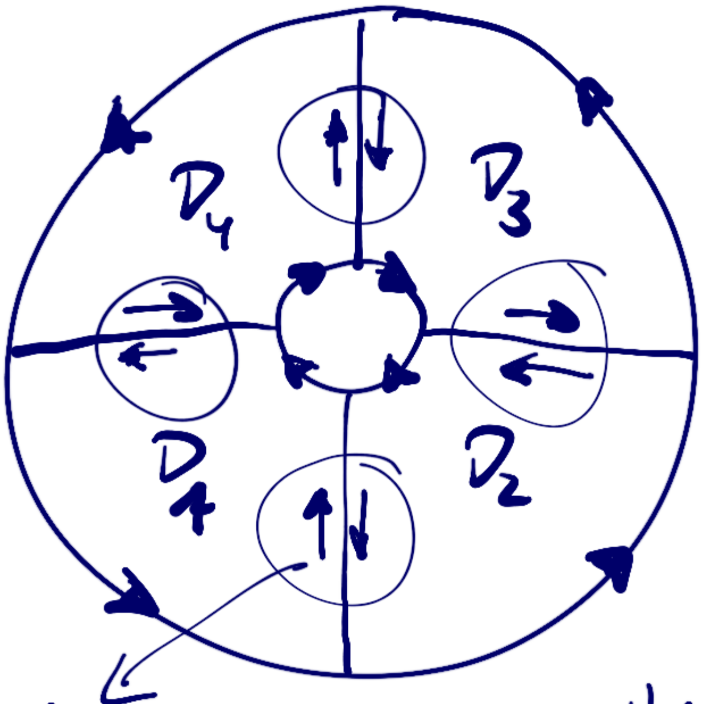
$$\int_C F\cdot dr = \underbrace{\int_{C_1} F\cdot dr} + \underbrace{\int_{C_2} F\cdot dr}.$$

How is the boundary?

$\Downarrow$

$C_1 \cup C_2$

To apply Green's Theorem we can divide the region.



$D = D_1 \cup D_2 \cup D_3 \cup D_4$

line integrals for those artificial lines cancel each other.

We can apply Green's Theorem for

D_1, D_2, D_3, D_4

and adding the result we will find what happens in D.

Example:

$$I = \int_{C} (x+2y)\,dx + (3x-y)\,dy$$

$$F(x,y) = (\underbrace{x+2y}_{P(x,y)}, \underbrace{3x-y}_{Q(x,y)})$$

$$C = \{(x,y) \in \mathbb{R}^2, \ x^2+4y^2=4\} \text{ ellipse.}$$

a) Parametrization

$$\left.\begin{array}{l} x = 2\cos t \\ y = \sin t \end{array}\right\} \quad t \in [0, 2\pi], \quad \sigma(t) = (2\cos t, \sin t)$$

$$I = \int_{C} F \cdot dr = \int_{0}^{2\pi} F(\sigma(t)) \cdot \sigma'(t)\,dt$$

$$F(\sigma(t)) = (2\cos t + 2\sin t, \ 6\cos t - \sin t)$$

$$\sigma'(t) = (-2\sin t, \cos t)$$

$$I = \int_0^{2\pi} \left(-5\cos t\sin t - 4\underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}} - 6\underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}}\right) dt$$

$$= 2\pi$$

b) Green's Theorem (quicker)

$$\underline{\underline{I}} = \int_C F \cdot dr = \iint_E \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_E (3-2)\,dx\,dy = \underline{\text{Area of Ellipse}}$$

$$= \boxed{a \cdot b\,\pi} = 2\pi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \sim \frac{x^2}{4} + y^2 = 1$$